Solitary Filaments of Coupled Electromagnetic Wave and Finite Amplitude Ion Fluctuations

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(Z. Naturforsch. **31 a,** 1517–1519 [1076]; received September 11, 1976)

We show analytically that the nonlinear coupling of a large amplitude electromagnetic wave with finite amplitude ion fluctuations leads to filamentation. The latter consists of striations of the electromagnetic radiation trapped in depressions of the plasma density. The filamentation is found to be either standing or moving normal to the direction of the incoming radiation. Criteria for the existence of localized filaments are obtained. Small amplitude results are discussed.

It is well-known [1–5] that an electromagnetic wave becomes modulationally unstable with respect to density fluctuations in a direction normal to the radiation. The corresponding filamentation instability leads to isolated one dimensional channels of electromagnetic radiation trapped in density cavities [1, 6]. In discussing the filamentary equilibria, the above mentioned papers assume that the density fluctuation δn is much smaller than the unperturbed density n_0.

In this paper, we investigate the nonlinear filamentation of a linearly polarized electromagnetic wave in a homogeneous plasma. The nonlinearity in the low-frequency ion perturbations is included in our formulation. A set of equations governing the coupling of the high-frequency electromagnetic waves with finite amplitude ion fluctuations are obtained. It is found that the coupled set of equations admit localized solutions. The criteria for the existence of the latter are presented. It is shown that a filament may be standing or moving with a finite speed, depending upon the magnitudes of the density depression and the high-frequency field. Small amplitude limits which lead to previously known results are discussed.

The equation for the electromagnetic wave electric field is given by

$$\partial^2 E/\partial t^2 - c^2 \nabla^2 E + c^2 \nabla \nabla \cdot E - \omega_p{}^2(z,t) E = 0\,, \qquad (1)$$

where $\omega_p{}^2 = (4 \pi n_0 e^2/m) n(z,t)/n_0$, and n_0 is the unperturbed density $n(\infty, t)$. The last term in (1) includes the nonlinear interaction of the radiation with plasma particles. We consider an envelope of linearly polarized electromagnetic wave propagating along the x axis, with its electric vector in the y direction. The amplitude E is taken to be a slowly varying function of z and t in order to account for the coupling with ion fluctuations. We have

$$E(x,z,t) = \hat{y}\, E(z,t) \exp\{i(k x - \omega t)\} + c \cdot c\,. \qquad (2)$$

In the usual WKB approximation, Eq. (1) may be written as [6]

$$i\, \varepsilon\, \partial E/\partial t + \partial^2 E/\partial z^2 - \Delta E = a^2 (n-1) E\,, \qquad (3)$$

where t, z, n, and E have been nondimensionalized by ω_{pi}^{-1}, λ_e, n_0, and $(4 \pi n_0 T_e)^{1/2}$ respectively. Here, $\lambda_e = v_{\text{Te}}/\omega_{\text{pe}}$, $\varepsilon = 2 (m_e/m_i)^{1/2} (\omega/\omega_{\text{pe}}) a^2$, $a = v_{\text{Te}}/c$, $\Delta = a^2 (a^2 k^2 + \omega_{\text{pe}}^2 - \omega^2)/\omega_{\text{pe}}^2$. We have also defined $v_{\text{tj}} = (T_j/m_j)^{1/2}$ and $\omega_{\text{pj}} = (4 \pi n_0 e^2/m_j)^{1/2}$.

Letting

$$E = \mathcal{E}(z - M\tau, \tau) \exp i[\Theta(\tau) + \Phi(z)] \qquad (4)$$

we obtain from (3)

$$-\varepsilon \frac{\partial \Theta}{\partial z} \frac{\partial^2 \mathcal{E}}{\partial z^2} - \Delta \mathcal{E} - \mathcal{E} (\partial \Phi/\partial z)^2 = a^2 (n-1) \mathcal{E} \qquad (5)$$

$$-\varepsilon M \frac{\partial \mathcal{E}}{\partial z} + 2 \left(\frac{\partial \Phi}{\partial z}\right) \frac{\partial \mathcal{E}}{\partial z} \frac{\partial^2 \Phi}{\partial z^2} \mathcal{E} = 0\,, \qquad (6)$$

where $M = v_0/c_s$ is the yet unknown Mach number.

From (6) we find

$$\Phi = \varepsilon M z/2\,. \qquad (7)$$

Inserting (7) into (5) leads to

$$\partial^2 \mathcal{E}/\partial z^2 - A\, \mathcal{E} = a^2 (n-1) \mathcal{E}\,, \qquad (8)$$

where $A = \Delta + \varepsilon (\partial \theta/\partial \tau) + \varepsilon^2 M^2/4$ is the nonlinear frequency shift.

The low-frequency density fluctuations respond to the wave electric field according to the fluid equations, namely

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial z} n_i v_i = 0 \qquad (9)$$

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$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial z} = - \frac{e}{m_i} \frac{\partial \Phi}{\partial z} \tag{10}$$

$$n_e = n_0 \exp(\varphi_s - \varphi_p) , \tag{11}$$

$$\partial^2 \Phi / \partial z^2 = 4 \pi e (n_e - n_i) \tag{12}$$

where $\varphi_s = e \Phi / T_e$ is the normalized ambipolar potential, and $\varphi_p = (m/T_e) < |\tilde{v}_e|^2 > = (\omega_{pe}^2/\omega^2)(E^2/4 \pi n_0 T_e)$ is the ponderomotive potential. The latter represents the momentum transfer between the high-frequency radiation and the plasma. The angular brackets denote the fast time average of the $\underset{\sim}{v}$, $\Delta \underset{\sim}{v}$ and $\underset{\sim}{v} \times \underset{\sim}{B}$ terms in the lectron momentum equation over the period $2\pi/\omega$. The radiation pressure mainly acts on the electrons because of the large ion mass. The electrons quickly return to equilibrium through the generation of an electrostatic ambipolar field which transfers the radiation pressure to the ions.

Making use of the charge neutrality condition $n_i = n_e$, we obtain

$$\varphi = \varphi_p + \ln n_i . \tag{13}$$

Substituting (13) into (10), and letting

$$n = n_i (z - M t) \tag{14}$$

$$v = v_i (z - M t) \tag{15}$$

we find

$$\frac{\partial}{\partial z} n (v - M) = 0 \tag{16}$$

$$\frac{1}{2} \frac{\partial}{\partial z} (v - M)^2 = - \frac{\partial}{\partial z} (\ln n + b\, \mathcal{E}^2) \tag{17}$$

where $b = \omega_{pe}^2 / 2 \omega^2$ and $v = v/c_s$.

Letting $u = v - M$, one finds from (16)

$$u = - M/n \tag{18}$$

where the plasma is taken to be unperturbed at $z = \pm \infty$, (i. e. $n = 1$, $v = 0$). Equation (17) can be integrated once to give

$$u^2/2 + \ln n + b\, \mathcal{E}^2 = C . \tag{19}$$

Using the boundary conditions at $z = \pm \infty$ yields $C = M^2/2$. Combining (18) and (19) we find

$$\mathcal{E}^2 = (M^2/2\, b)(1 - n^{-2}) - b^{-1} (\ln n) . \tag{20}$$

We now impose the boundary conditions $n = N$, $\mathcal{E} = \mathcal{E}_0$ at $z = 0$, the Mach number M is then uniquely determined from (20). We obtain

$$M^2 = 2 [\ln N + b\, \mathcal{E}_0^2] / (1 - N^{-2}) . \tag{21}$$

Multiplying (8) by $\partial \mathcal{E}/\partial z$, using (20) and integrating the resulting equation with respect to z, we find

$$(\partial \mathcal{E}/\partial z)^2 + (a^2 - A)\, \mathcal{E}^2 + (a^2/b)\, n (1 + M^2/n^2) - (a^2/b)(1 + M^2) = 0 \tag{22}$$

where again we have made use of the boundary conditions at $z = \pm \infty$ to obtain the integration constant.

Noting that at $z = 0$, $n = N$, $\mathcal{E} = \mathcal{E}_0$, $\partial \mathcal{E}/\partial z = 0$, we readily find from (22) the nonlinear frequency shift in terms of $\mathcal{E}_0$ and N,

$$A = a^2 \left[1 - \frac{2 (b\, \mathcal{E}_0^2 + \ln N) N}{(N+1)\, b\, \mathcal{E}_0^2} + \frac{N-1}{b\, \mathcal{E}_0^2} \right] . \tag{23}$$

From (20), we have

$$\left(\frac{\partial n}{\partial z} \right)^2 = \frac{4 n^2 \mathcal{E}^2}{(M^2/n^2 - 1)^2} \left(\frac{\partial \mathcal{E}}{\partial z} \right)^2 . \tag{24}$$

Combining (22) and (24), one obtains

$$(\partial n/\partial z)^2 + V(n, N, \mathcal{E}, b) = 0 , \tag{25}$$

where

$$V(n, N, \mathcal{E}, b) = \frac{4 a^2 n^2 \mathcal{E}^2}{b (M^2/n^2 - 1)^2} \left[(1 - A/a^2)\, b\, \mathcal{E}^2 + (n-1) \left(1 - \frac{M^2}{n} \right) \right] , \tag{26}$$

and $\mathcal{E}$ is given by (20).

It follows from (25) that localized solutions exist provided that $V < 0$. For this reason, we analyze (26) near $n = 1$ and $n = N$ and see whether V behaves properly at the boundaries. We have

$$V(n) = \begin{cases} -4 (1-n)^2 A/b^2 \quad \text{for } n = 1 , & (27) \\ \dfrac{4 N^2 \mathcal{E}_0^2 a^2 (n - N)}{b (1 - M^2/N^2)} [1 - (1-N)/b N \mathcal{E}_0^2) (1 - M^2/N)] . & (28) \\ \text{for } n \cong N & \end{cases}$$

Equations (27) and (28) show that V is negative for $A \geqq 0$, and $N^2 > M^2$.

In Fig. 1, we have indicated the region of existence of soliton solutions. Lines of constant soliton velocity are plotted for several Mach numbers. From the Figure we have the following conclusions: Firstly, for a fixed density depression, the field intensity varies oppositely to the filament velocity. This statement follows from (17) which corresponds to an energy conservation law if $(1 - N^2)/N^2$ is interpreted as an effective mass, $\mathcal{E}^2$ as the potential energy,

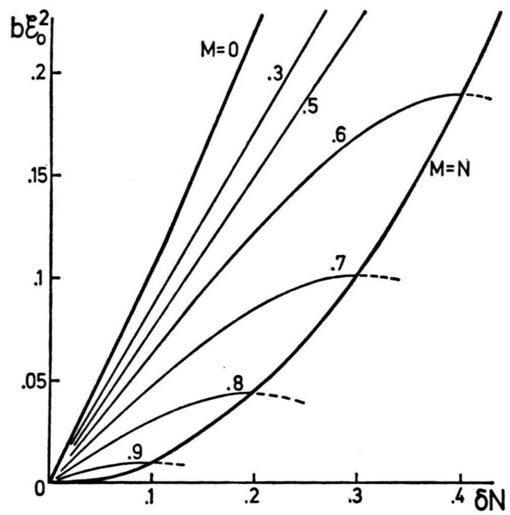

Fig. 1. The existence region of localized solutions. Here, $\mathcal{E}_0^2 = |\mathcal{E}|^2_{max}/4\pi n_0 T_e$, $\delta N = 1 - N (= n_{min}/n_0)$. Localized filaments exist between the lines $M=0$ and $M=N$.

and $-\ln N$ as the total energy of the system. A standing envelope filament has therefore maximum field intensity. Secondly, Fig. 1 indicates that high speed ($M \leqq 1$) envelope filaments have always small amplitudes.

Let us briefly derive the small amplitude limit. Letting $(\delta N)^2 \ll b\,\mathcal{E}_0^2 \lesssim \delta N \ll 1$, where $\delta N = 1 - N$, we find

$$V(\delta n, \delta N, W^2) = -(2\,a^2\,\delta n/b^2)\,(\delta N - \delta n) \tag{29}$$

where $\delta n = 1 - n$, and

$$M^2 = 1 - \frac{b\,\mathcal{E}_0^2}{\delta N} - \delta N\left(1 - \frac{3}{2}\,\frac{b\,\mathcal{E}_0^2}{\delta N}\right), \tag{30}$$

$$A = \frac{a^2\,\delta N}{2}\left[1 + \frac{\delta N}{2}\left(1 - \frac{2}{3}\,\frac{\delta N}{b\,\mathcal{E}_0^2}\right)\right]. \tag{31}$$

The density depression and the electric field of the envelope is given by

$$\delta n = \delta N\,\text{sech}^2\,[(\delta N/6)^{1/2}\,z]\,, \tag{32}$$

$$\mathcal{E}^2 = b\,\mathcal{E}_0^2\,\text{sech}^2\,[(\delta N/6)^{1/2}\,z]\,, \tag{33}$$

and $\Theta(t) = (A - \varepsilon^2 M^2/3)\,t/2\,\varepsilon$.

Thus, for $b\,\mathcal{E}_0^2/\delta N \lesssim 1$, we have subsonic envelope solutions [1]. On the other hand, for $\delta N^2 \ll b\,\mathcal{E}_0^2 \ll \delta N$ [e.g. $b\,\mathcal{E}_0^2 \approx 0(\delta N^{3/2})$] we recover sonic solutions presented by Karpman [1]. We recall that near sonic solutions obtained by Kaw and Nishikawa [6] obeys the ordering $b\,\mathcal{E}_0^2 = 0\ (\delta N^2)$. For this case, the use of quasi-neutrality can be questioned. This investigation is beyond the scope of this paper. Finally, we mention that finite amplitude Langmuir solitons [7], as well as envelope whistler solitons [8] have been treated elsewhere.

The authors wish to acknowledge the benefit of useful discussions with Hans Schamel. This work has been performed under the auspices of the Sonderforschungsbereich 162 "Plasmaphysik Bochum/Jülich". SGT is grateful to the Alexander von Humboldt Foundation for the award of a Senior Fellowship.

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